AD-A100 003 MARYLAND UNIV COLLEGE PARK COMPUTER VISION LAB F/G 12/1

SOME RESULTS ON FUZZY (DIGITAL) CONVEXITY.(U)

MAR 81 L JANOS, A ROSENFELD AFOSR-77-3271

UNCLASSIFIED TR-1035 AFOSR-TR-81-0469 NL

1 of 1

AD A100003

END

DATE FILMED

6 81

DTIC

AFOSR-TR- 81-0469

LEVEL II

AD A100003

TR-1035
AFOSR-77-3271

Mar[redacted]981

SOME RESULTS ON FUZZY (DIGITAL) CONVEXITY

Ludvik Janos
Azriel Rosenfeld

Computer Vision Laboratory
Computer Science Center
University of Maryland
College Park, MD 20742

ABSTRACT

The definition of fuzzy convexity is reviewed, and some results on projections of convex and fuzzy-convex sets are established. Digital fuzzy convexity is defined, and relationships among alternative definitions are investigated.

The support of the U.S. Air Force Office of Scientific Research under Grant AFOSR-77-3271 is gratefully acknowledged, as is the help of Sherry Palmer in preparing this report.

81 6 10 041

1. Introduction

Convexity was one of the first mathematical concepts to be "fuzzified" when fuzzy set theory was initially developed [1]. This paper reviews the concept and presents some results on projections of convex and fuzzily convex sets. It also introduces the concept of digital fuzzy convexity for sets of lattice points; this is a "fuzzification" of digital convexity, which has been extensively studied [2]. We deal here with (fuzzy) subsets of the plane, or with planar lattice points, even though many of the concepts introduced have immediate extensions to higher-dimensional spaces.

2. Fuzzy convexity

2.1 Basic concepts

Let E be the Euclidean plane; we recall [1] that a fuzzy subset μ of E is a mapping from E into [0,1]. The value $\mu(P)$ of μ for a point $P \in E$ is called the degree of membership of P in μ. We say that μ is (fuzzily) convex if for all P,Q in E, and all R on the line segment $\overline{PQ}$, we have $\mu(R) \geq \mu(P) \wedge \mu(Q)$, where $\wedge$ means "min" - in other words, every point on $\overline{PQ}$ has degree of membership in μ at least as high as the lesser of the degrees of P and Q.

If μ is a mapping into {0,1}, it is the characteristic function of a subset of E (namely, $\mu^{-1}(1)$); for brevity, we speak of such a μ as "being" a subset of E. Evidently, a subset μ is fuzzily convex iff it is convex in the ordinary sense. Indeed, if μ is into {0,1}, the condition $\mu(R) \geq \mu(P) \wedge \mu(Q)$ is vacuous unless $\mu(P) = \mu(Q) = 1$, and it then requires that $\mu(R) = 1$; in other words, if P,Q are in μ, it requires that any point on the segment $\overline{PQ}$ also be in μ, which is the standard definition of convexity.

A real-valued function f defined on the real line will be called min-free if, for all points $A \leq B \leq C$, we have $f(B) \geq f(A) \wedge f(C)$. Thus a fuzzy set μ is convex iff all its cross-sections are min-free functions. In Sections 2.2-3 we will consider under what circumstances the projections of (fuzzily) convex sets are min-free functions. Note that a fuzzy subset of the real line is convex iff, regarded as a real-valued function, it is min-free.

For any $0 \leq t \leq 1$, the set $\mu_t \equiv \{P \in E | \mu(P) > t\}$ will be called a level set of μ.

Proposition 1. μ is convex iff its level sets are all convex. (We regard the empty set as convex.)

Proof: $\mu(P)$ and $\mu(Q) > t$ require $\mu(R) > t$ for all $R \in \overline{PQ}$, making μ_t convex. Conversely, given any P,Q and any $R \in \overline{PQ}$, suppose $\mu(R) < \mu(P) \wedge \mu(Q)$, and let $t = \mu(R)$; then $\mu(P) > t$ and $\mu(Q) > t$, so that μ_t is not convex.//

Readily, the proposition is also true if we define "level set" using $\geq$ instead of $>$.

2.2 The sup projection

For any line ℓ and any point $P \in \ell$, let ℓ_P be the line perpendicular to ℓ at P. By the sup projection of a fuzzy set μ on ℓ we mean the function μ_ℓ that maps each point $P \in \ell$ into $\sup\{\mu(Q) \mid Q \in \ell_P\}$. Evidently μ_ℓ is a fuzzy subset of ℓ, since $0 \le \mu_\ell \le 1$, and if μ is an ordinary set, so is μ_ℓ for all ℓ. It is easily seen that if μ is a connected set, μ_ℓ is an interval. (Indeed, given any $P, Q \in \mu_\ell$, there must exist points of μ on ℓ_P and ℓ_Q; since μ is connected, there is a path in μ joining these points, and readily the sup projection of this path must contain the interval $\overline{PQ}$.)

Proposition 2. If μ is convex, so is μ_ℓ.

Proof: Let A,B,C (in that order) be points of ℓ. Given any $\varepsilon > 0$, let A' and C' be points on ℓ_A and ℓ_C, respectively, such that $\mu_\ell(A) < \mu(A') + \varepsilon$ and $\mu_\ell(C) < \mu(C') + \varepsilon$. Let B' be the intersection of segment $\overline{A'C'}$ with ℓ_B. Since μ is convex and $B' \in \overline{A'C'}$, we have

$$\mu(B') \ge \mu(A') \wedge \mu(C') = [\mu_\ell(A) - \varepsilon] \wedge [\mu_\ell(C) - \varepsilon] = [\mu_\ell(A) \wedge \mu_\ell(C)] - \varepsilon$$

But $\mu(B') \le \mu(B)$ by definition of the sup projection. Hence $\mu_\ell(B) \ge [\mu_\ell(A) \wedge \mu_\ell(C)] - \varepsilon$, and since ε is arbitrary, we have $\mu_\ell(B) \ge \mu_\ell(A) \wedge \mu_\ell(C)$, proving μ_ℓ convex.//

The converse of Proposition 2 is false; even if all the sup projections of μ are convex, μ need not be convex. To see this, let μ be an ordinary set and suppose that μ is connected. By the remarks preceding Proposition 2, the sup projection of μ on any ℓ is an interval, hence is convex, but μ itself need not be convex.

2.3 The integral projection

By the integral projection of μ on ℓ we mean the function $\bar{\mu}_\ell$ that maps each point $P\in\ell$ into $\int_{\ell_P}\mu$, the integral of μ over the line ℓ_P perpendicular to ℓ at P. Note that if μ is an ordinary convex set, ℓ_P meets μ in an interval, and $\int_{\ell_P}\mu$ is just the length of this interval. We assume here that this integral always exists. Note that we no longer have $0\le\bar{\mu}_\ell\le 1$, as we did in the case of the sup projection.

Proposition 3. If μ is a convex set, $\bar{\mu}_\ell$ is a min-free function.

Proof: Let A,B,C, (in that order) be points of ℓ. Each of the lines ℓ_A,ℓ_B,ℓ_C meets the convex set μ in an interval (possibly degenerate or empty);
let the endpoints of these intervals be A',A",B',B", and C',C", respectively (see Figure 1).
Since μ is convex, the segments $\overline{A'C'}$ and $\overline{A''C''}$ are subsets of μ; hence the points P,Q where these segments meet ℓ_B are in μ, and lie between B' and B". Now evidently $\min(|A'A''|,|C'C''|)\le |PQ|\le\max(|A'A''|,|C'C''|)$, where bars denote the length of an interval. But $|A'A''|=\bar{\mu}_\ell(A)$ and $|C'C''|=\bar{\mu}_\ell(C)$, as pointed out in the preceding paragraph. Hence $\bar{\mu}_\ell(B)=|B'B''|\ge|PQ|\ge\min(|A'A''|,|C'C''|)=\bar{\mu}_\ell(A)\wedge\bar{\mu}_\ell(C)$, proving that $\bar{\mu}_\ell$ is min-free.//

Unfortunately, Proposition 3 is false if μ is only assumed to be fuzzily convex. To see this, let μ be defined as follows: $\mu = 0.1$ in the quadrilateral whose vertices are (0,0),(0,10), (9,0), and (9,1); except that $\mu = 0.9$ on the line segment $\overline{(9,0),(9,1)}$ (see Figure 2). Since the level sets of μ are convex, μ is fuzzily convex (see Section 2.1). Fut for the integral projection of μ on the x-axis we have $\overline{\mu}(0) = 10$, $\overline{\mu}(9) = 9$, while $\overline{\mu}(5) = 5$, so that $\overline{\mu}$ is not a min-free function.

The converse of Proposition 3 is also false; even if all the integral projections of μ are min-free functions, μ is not necessarily convex. In fact, consider the L-shaped polygon Π whose vertices are (0,0),(0,2),(2,0),(1,2),(1,1), and (2,1) (see Figure 3), and project Π onto an arbitrary line ℓ (Figure 4). It is evident that the value of this projection $\overline{\mu}$ has no strict local minimum (see Figure 4: it strictly increases from P_1 to P_2, remains constant from P_2 to P_3, strictly decreases from P_3 to P_4, remains constant from P_4 to P_5, and strictly decreases from P_5 to P_6), hence is a min-free function, but Π, of course, is not convex.

3. Fuzzy digital convexity

3.1 Digital convexity [2]

Let R be a subset of the plane such that $\overline{(R^0)}=R$ (R is the closure of its interior); we call such an R regular. Let us regard each lattice point P as the center of an open unit square (a "cell") P^*. The set $I(R)\equiv\{P|R\cap P^*\neq\emptyset\}$ is called the digital image of R. Note that we have not defined the digital image for arbitrary sets, but only for regular sets.

Proposition 4. $R\subseteq\cup\{\overline{P}^*|P\in I(R)\}$, and I(R) is the smallest set of lattice points for which this is true.

Proof: By definition of I(R), R meets Q^* iff $Q\in I(R)$; and if R meets any $\overline{Q}^*$ on its boundary, it meets the interior of at least one of the cells that share that boundary.//

A set of S of lattice points is called digitally convex if it is the digital image of a convex regular set R.

Proposition 5. A digitally convex set is 4-connected.

Proof: We show that the digital image S of any arcwise connected regular set R is 4-connected. For all $P,Q\in S$, R meets P^* and Q^*, say in the points (x,y) and (u,v), and there is a path in R from (x,y) to (u,v). It is easily seen that this path meets a sequence of interiors of 4-adjacent cells which thus yield a 4-path in S from P to Q.//

The proofs of the following two theorems can be found in [2].

Theorem 6. The following properties of a 4-connected set S are equivalent:

(a) For all P,Q, in S, no point not in S lies on the line segment $\overline{PQ}$

(b) For all P,Q, in S, and all $(u,v)\in\overline{PQ}$, there exists a point $(x,y)\in S$ such that $\max(|x-u|,|y-v|)<1$.//

We call S regular if every $P\in S$ has at least two (horizontal or vertical) neighbors in S.

Theorem 7. Any digitally convex set has the properties of Theorem 6. A regular set S is digitally convex iff it has the properties of Theorem 6.//

If S is not regular, it may satisfy the properties of Theorem 6 but not have a convex preimage.

In Section 3.2 we discuss the possibility of generalizing these results to fuzzily convex sets.

3.2 Fuzzy digital convexity

Given a fuzzy subset μ of the plane, we define a fuzzy subset μ' of the lattice points by

$$\mu'(P) \equiv \sup\{\mu(x,y) \mid (x,y) \in P^*\}$$

Proposition 8. If μ_t is regular, μ'_t is its digital image.

Proof: μ_t meets P^* iff $\sup\{\mu(x,y) \mid (x,y) \in P^*\} > t$ iff $\mu'(P) > t$ iff $P \in \mu'_t$.

The corresponding statement is not true if we use $\geq$ rather than $>$ in defining level sets. Indeed, if such a level set μ_t meets P^*, we have $\sup\{\mu(x,y) \mid (x,y) \in P^*\} \geq t$, so that $\mu'(P) \geq t$ and $P \in \mu'_t$; but conversely, if the $\sup \geq t$, μ_t may only meet $\overline{P}^*$ (though it does have to meet the interior of some cell that shares its border with P^*, if μ_t is regular). Thus we know only that if μ_t is regular μ'_t contains its digital image.

Corollary 9. If μ is an ordinary regular set, μ' is its digital image.

Proof: $\mu = \mu_0$ is regular, hence $\mu'_0 = \mu'$ is its digital image.

We call μ fuzzily regular if all its level sets μ_t are regular, $0 \leq t < 1$. If μ is fuzzily regular, we call μ' its digital image.

We call μ' fuzzily digitally convex (FDC) if it is the digital image of a fuzzily regular, fuzzily convex μ.

Analogous to Proposition 1 we then have

Proposition 11. If μ' is FDC, all its level sets are digitally convex.

Proof: Every μ_t' is the digital image of μ_t (Proposition 8), which is convex (Proposition 1).//

Analogous to Condition (a) in Theorem 6, we have

Proposition 12. If μ' is FDC, then for all collinear triples of lattice points A,B,C, with B between A and C, we have $\mu'(B) \geq \mu'(A) \wedge \mu'(C)$.

Proof: Given any $\varepsilon > 0$, let A',C' be points of the cell interiors A*,C* such that $\mu'(A) \leq \mu(A') + \varepsilon, \mu'(C) \leq \mu(C') + \varepsilon$, where $\mu' = I(\mu)$. Evidently, $\overline{A'C'}$ meets the cell interior B*; let B' be a point of $B^* \cap A'C'$. Since μ is fuzzily convex, we have $\mu(B') \geq \mu(A') \wedge \mu(C') > (\mu'(A) - \varepsilon) \wedge (\mu'(C) - \varepsilon) = (\mu'(A) \wedge \mu'(C)) - \varepsilon$. Since $\mu'(B) = \sup\{\mu(x,y) \mid (x,y) \in B^*\} \geq \mu(B')$, we thus have $\mu'(B) > \mu'(A) \wedge \mu'(C) - \varepsilon$; and since ε is arbitrary, it follows that $\mu'(B) \geq \mu'(A) \wedge \mu'(C)$.

References

1. L.A. Zadeh, Fuzzy sets, Information and Control 8, 1965, 338-353.

2. C.E. Kim, On the cellular convexity of complexes, IEEE Transactions on Pattern Analysis and Machine Intelligence, to appear.

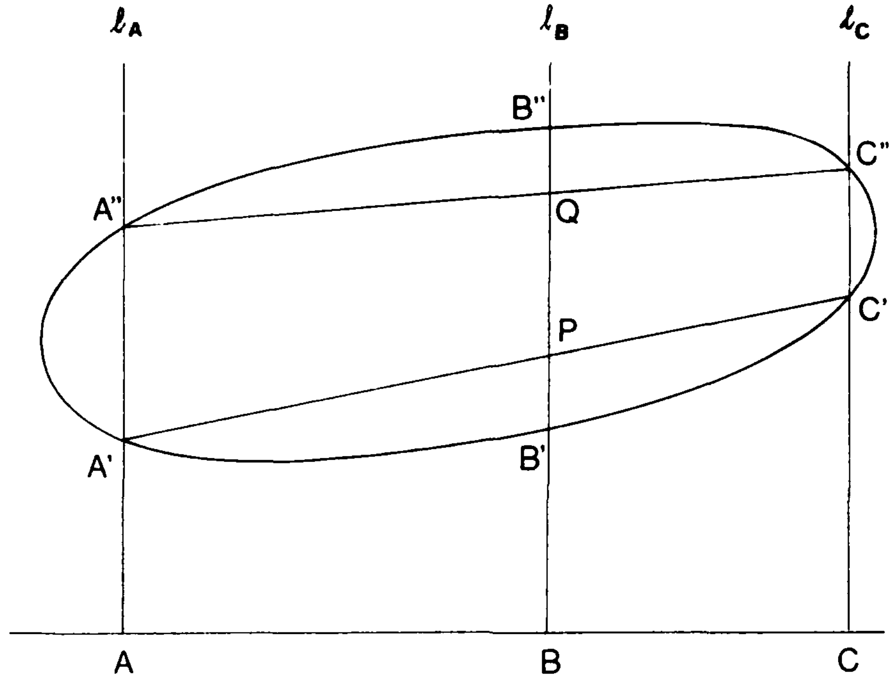

Figure 1

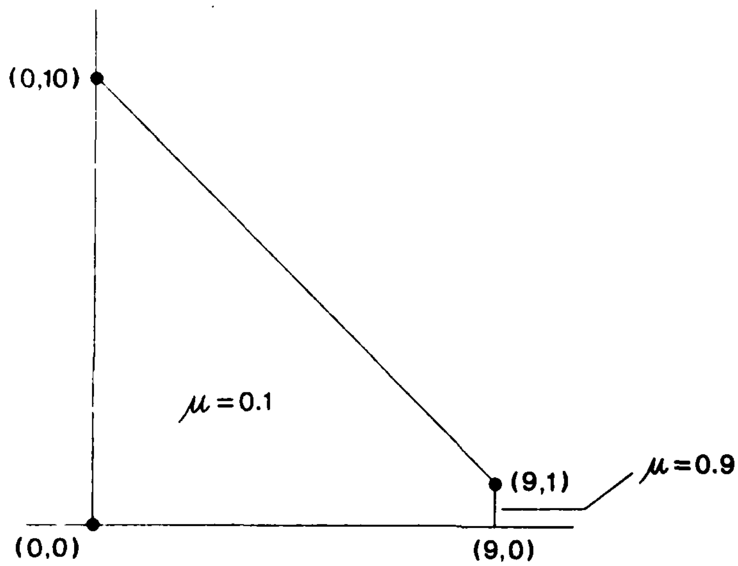

Figure 2

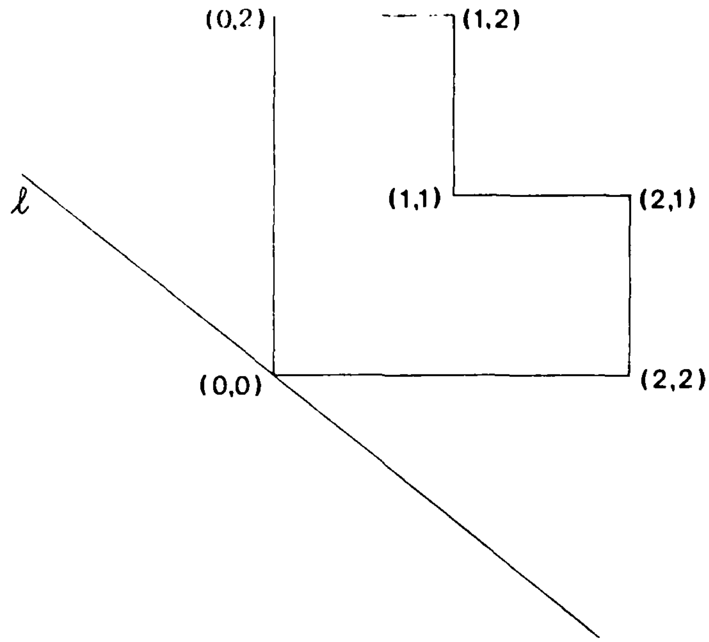

Figure 3

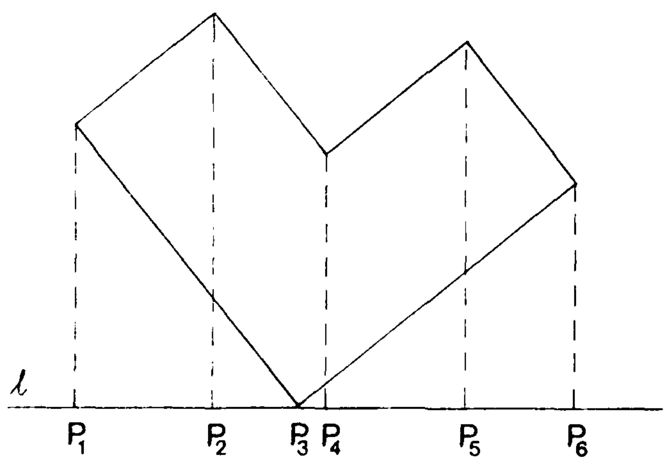

Figure 4

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 81-0469	2. GOVT ACCESSION NO. AD-A100 003	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME RESULTS ON FUZZY (DIGITAL) CONVEXITY		5. TYPE OF REPORT & PERIOD COVERED INTERIM
		6. PERFORMING ORG. REPORT NUMBER TR-1035
7. AUTHOR(s) L. Janos, A. Rosenfeld		8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3271
9. PERFORMING ORGANIZATION NAME AND ADDRESS Computer Science Center, University of Maryland, College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE 61102F 2304/A2
11. CONTROLLING OFFICE NAME AND ADDRESS Math & Info. Sciences, AFOSR/NM, Bolling AFB, Washington, DC 20332		12. REPORT DATE March 1981
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Image processing
Pattern recognition
Digital geometry
Convexity
Fuzzy sets

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The definition of fuzzy convexity is reviewed, and some results on projections of convex and fuzzy-convex sets are established. Digital fuzzy convexity is defined, and relationships among alternative definitions are investigated.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)